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Spheres with varying density in general relativity

M. C. DURGAPAL and G. L. GEHLOT
Department of Physics, University of Jodhpur, Jodhpur, India
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Abstract. Exact internal solutions for massive spheres with three different distributions of density have been found. The central core density is considered to be constant while the outer region has two types of varying densities: (i) varying inversely as the square of the radial distance; (ii) varying parabolically as $\rho \propto 1 - r^2/\text{constant}$. The solutions are found to be continuous at different boundaries while the densities of different regions are not continuous but vary slightly. With proper choice of different parameters appearing in the solutions, we can apply the results to actual problems of stellar physics.

1. Introduction

In actual problems related to stellar bodies the exact internal solutions of spheres with constant density can not be applied, because the density does not remain constant throughout the sphere. The authors (1969) have found exact internal solutions for dense massive stars with two density distributions. The only drawback is that the variation of the density in the outer region is restricted only to a particular distribution of matter. The variation of the density may not be the same in the whole outer region. It may be slow in the beginning and fast later or it may be fast in the beginning and slow later. Harrison *et al.* (1965) solved the problem by considering a scaling law and dividing the whole sphere in three different zones. Various parameters were calculated by numerical methods.

In this paper the outer region is assumed to have two different types of density variations: (i) $\rho \propto 1/r^2$ and (ii) $\rho \propto (1 - r^2/\text{constant})$. An inverse square variation and a parabolic variation of density are chosen. Particular solutions of Einstein's field equations have been found exactly involving no computer calculations. The constants of proportionality can be chosen to suit various types of models for stellar bodies.

In choosing these density distributions we have taken into account the following points:

(i) The density variation $\rho \propto 1/r^2$ is the most rapid variation of density. If the power of r becomes less than -2 the solutions obtained are physically unreasonable. Moreover, in this choice of density e^λ comes out as a constant for the whole region. e^λ appears in the expressions for proper mass and proper volume while solving the problems related to gravitational collapse.

(ii) The variation $\rho \propto 1 - r^2/K^2$ is the most smooth variation of density. The density variation reduces as the value of K increases. When $K = \infty$ the density becomes constant and the solutions reduce to those of a constant density sphere.

The general assumptions made here for solving the Einstein's field equations are the following:

(i) The density distribution: In part (2) of the solutions the density is constant $= \rho_0$ in the central core $0 \leqslant r \leqslant r_1$; in the middle region $r_1 \leqslant r \leqslant r_2$ the density is given by

$$\rho = \rho'\left(1 - \frac{r^2}{K^2}\right) \tag{1}$$

where $K \geqslant r_2$, and in the outermost strata the density is

$$8\pi\rho = C/r^2. \tag{2}$$

In part (1) of the solutions the density is constant and equal to ρ_c, in the central core $0 \leqslant r \leqslant r_1$; in the middle region $r_1 \leqslant r \leqslant r_2$

$$8\pi\rho = C/r^2 \tag{2'}$$

and in the outermost strata $r_2 \leqslant r \leqslant r_3$

$$\rho = \rho'\left(1-\frac{r^2}{K^2}\right) \tag{1'}$$

where $K \geqslant r_3$.

(ii) The system is spherically symmetric and static with regular space–time, the centre of symmetry being the origin.

(iii) The space is empty outside a finite region of radius r_3. At $r = r_3$ the internal solutions and external Schwarzschild solutions have the same value. This is necessary for continuity.

(iv) The solutions in all the three regions must have the same value at the internal boundaries $r = r_1$ and $r = r_2$.

(v) The pressure must be continuous at the internal boundary and must vanish at the surface $r = r_3$.

(vi) The pressure and density must be finite and positive at all the points, with the restrictions

$$P \leqslant \tfrac{1}{3}\rho \text{ (Bondi 1964)} \qquad \text{or} \qquad P \leqslant \rho \text{ (Zeldovich 1962).}$$

Taking the velocity of light $c = 1$ and the gravitational constant $G = 1$, the relations between density and the pressure P and the components of the energy–momentum tensor of a perfect fluid are given by

$$\rho = T_0{}^0 \qquad -P = T_1{}^1 = T_2{}^2 = T_3{}^3 \qquad T_\nu{}^\mu = 0 \qquad (\mu \neq \nu)$$

2. Field equations and their solutions

The line element is given by

$$ds^2 = e^\nu\, dt^2 - e^\lambda\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2. \tag{4}$$

Here λ and ν are functions of r alone. The resulting field equations (Tolman 1934) are

$$-8\pi T_1{}^1 = 8\pi P = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 \tag{5}$$

$$-8\pi T_2{}^2 = -8\pi T_3{}^3 = 8\pi P = e^{-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\lambda'\nu'}{4}+\frac{\nu'-\lambda'}{2r}\right) \tag{6}$$

$$-8\pi T_0{}^0 = -8\pi\rho = -1/r^2 - e^{-\lambda}(\lambda'/r - 1/r^2). \tag{7}$$

Part (1): Outer region $r_2 \leqslant r \leqslant r_3$:

The equations (1′) and (7) give

$$e^{-\lambda} = 1 - \frac{8\pi\rho'}{15}\left(5r^2 - \frac{3r^4}{K^2}\right) + \frac{\text{constant}}{r} \qquad K \geqslant r_3$$

putting the constant of integration zero for simplification. We could have started with the assumption that $e^{-\lambda} = 1 - \text{const} \times r^2 - \text{const} \times r^4$ and obtained the expression for density, because it is e^{λ} which appears in all the equations of proper volume or proper mass. So in the present paper we shall take the integration constant in the solution of $e^{-\lambda}$ to be zero. We can further write the expression for $e^{-\lambda}$ as

$$e^{-\lambda} = 1 - \frac{8\pi\rho' K^2}{15}(5x - 3x^2) \tag{8}$$

where $x = r^2/K^2$. The value of $e^{-\lambda}$ at $r = r_3$ is

$$e^{-\lambda} = 1 - 2m/r_3 \tag{9}$$

where 'm' is the total mass of the sphere. The continuity of $e^{-\lambda}$ at the external boundary gives us

$$m = \frac{4\pi\rho' r_3^3}{15}(5 - 3r_3^2/K^2). \tag{10}$$

Putting $\nu = 2 \ln u$ and using equations (5) and (6) we get

$$\frac{d^2u}{dw^2} + \frac{u}{4} = 0 \tag{11}$$

where

$$w = \ln\left\{x - \tfrac{5}{6} + \left(x^2 - \tfrac{5}{3}x + \frac{5}{8\pi K^2\rho'}\right)^{1/2}\right\}$$

giving us

$$u = C_1 \cos \tfrac{1}{2}w + C_2 \sin \tfrac{1}{2}w$$

or

$$e^{\nu} = u^2 = (C_1 \cos \tfrac{1}{2}w + C_2 \sin \tfrac{1}{2}w)^2 \tag{12}$$

where C_1 and C_2 are constants of integration.

From equations (5), (8) and (12) we get

$$8\pi P = M \frac{C_2 - C_1 \tan \frac{1}{2}w}{C_1 + C_2 \tan \frac{1}{2}w} - N \tag{13}$$

where

$$M = \left(\frac{32\pi\rho'}{5K^2}\right)^{1/2} \left(1 - \frac{8\pi K^2\rho'}{15}(5x - 3x^2)\right)$$

and

$$N = \frac{8\pi\rho'}{15}(5 - 3x).$$

At the external boundary $r = r_3$ the pressure is zero and the value of e^{ν} is given by

$$e^{\nu} = 1 - 2m/r_3 = Q \text{ (say)}. \tag{14}$$

The vanishing of the pressure and the continuity of e^{ν} at the external boundary gives

$$C_1 = \frac{Q^{1/2}}{M_3}(M_3 \cos \tfrac{1}{2}w_3 - N_3 \sin \tfrac{1}{2}w_3) \tag{15}$$

$$C_2 = \frac{Q^{1/2}}{M_3}(M_3 \sin \tfrac{1}{2}w_3 + N_3 \cos \tfrac{1}{2}w_3) \tag{16}$$

where M_3, N_3 and w_3 are respective values of M, N and w when $x = r_3^2/K^2$.

The middle region $r_1 \leqslant r \leqslant r_2$:

Equations (2) and (7) give

$$e^{-\lambda} = 1 - C \tag{17}$$

the constant of integration being taken as zero. Equations (5), (6) and (17) lead to the equation

$$r^2 y'' - r y' - (n+1)(n-1) y = 0 \tag{18}$$

where

$$\nu = 2 \ln y \quad \text{and} \quad C = (1-n^2)/(2-n^2).$$

n is always less than 1 since C must always be less than one.

Equation (18) can be solved easily to give

$$e^{\nu} = (D_1 r^{1+n} + D_2 r^{1-n})^2 \tag{19}$$

where D_1 and D_2 are two constants of integration. The expression for the pressure in this region is obtained from equations (5), (17) and (19) as

$$8\pi P = \frac{2(1-C)}{r}\left(\frac{D_1(1+n)r^n + D_2(1-n)r^{-n}}{D_1 r^{1+n} + D_2 r^{1-n}}\right) - \frac{C}{r}. \tag{20}$$

The continuity of e^{λ}, e^{ν} and P at the boundary $r = r_2$ gives

$$C = 2m'/r_2 = \frac{8\pi\rho' r_2{}^2}{15}(5 - 3r_2{}^2/K^2) \tag{21}$$

$$D_1 r_2{}^{1+n} + D_2 r_2{}^{1-n} = C_1 \cos\tfrac{1}{2}w_2 + C_2 \sin\tfrac{1}{2}w_2 \tag{22}$$

$$\frac{2(1-C)}{r_2}\left(\frac{D_1(1+n)r_2{}^{1+n} + D_2(1-n)r_2{}^{1-n}}{D_1 r_2{}^{1+n} + D_2 r_2{}^{1-n}}\right) - \frac{C}{r_2^2} = M_2 \frac{C_2 - C_1 \tan\frac{1}{2}w_2}{C_1 + C_2 \tan\frac{1}{2}w_2} - N_2 \tag{23}$$

where m' = mass contained within the radius $r = r_2$ or

$$\frac{(1-n^2) r_2}{2(2-n^2)} = \int_0^{r_2} 4\pi\rho' \left(1 - \frac{r^2}{K^2}\right) r^2 \, dr. \tag{24}$$

Equation (24) decides the value of n to be chosen in the middle region in terms of ρ', r_2 and K. The equations (22) and (23) can be solved to get the exact values of D_1 and D_2. Thus we know all the constants appearing in the exact solutions of the middle region.

The central core $0 \leqslant r \leqslant r_1$: From the Schwarzschild solution for a constant density sphere (Tolman 1934)

$$e^{-\lambda} = 1 - r^2/R^2 \tag{25}$$

$$e^{\nu} = \{A - B(1 - r^2/R^2)^{1/2}\}^2 \tag{26}$$

$$8\pi P = \frac{1}{R^2}\left(\frac{3B(1-r^2/R^2)^{1/2} - A}{A - B(1-r^2/R^2)^{1/2}}\right) \tag{27}$$

where

$$R^2 = 3/8\pi\rho_c. \tag{28}$$

The continuity of $e^{-\lambda}$, e^{ν} and P at $r = r_1$ gives us

$$\frac{8\pi\rho_c r_1^2}{3} = C = \frac{(1-n^2)}{(2-n^2)} = \frac{8\pi\rho' r_2^2}{15}(5-3r_2^2/K^2). \tag{29}$$

$$A = \tfrac{1}{2}e^{\nu_1/2}(3+8\pi P_1 R^2) \tag{30}$$

$$B = \tfrac{1}{2}e^{\nu_1/2}(1+8\pi P_1 R^2)(1-r_1^2/R^2)^{-1/2} \tag{31}$$

where

$$e^{\nu_1} = (D_1 r_1^{1+n} + D_2 r_1^{1-n})^2$$

$$8\pi P_1 = \frac{2(1-C)}{r_1}\left(\frac{D_1(1+n)r_1^{n} + D_2(1-n)r_1^{-n}}{D_1 r_1^{1+n} + D_2 r_1^{1-n}}\right) - \frac{C}{r_1}.$$

Equation (29) is of vital importance as it correlates the central density ρ_c, the parameter n and ρ' in terms of radii r_1, r_2 and parameter K.

Part (2): The density is constant in the central region $0 \leqslant r \leqslant r_1$. In the outer region the density distribution is

$$8\pi\rho = C/r^2 \qquad r_2 \leqslant r \leqslant r_3$$

while in the middle region $r_1 \leqslant r \leqslant r_2$ the density is

$$\rho = \rho'(1-r^2/K^2).$$

We will not go into the details of the solutions and shall write only the relevant equations.

Outermost region $r_2 \leqslant r \leqslant r_3$:

$$e^{-\lambda} = 1-C = 1-2m/r_3 = \frac{1}{2-n^2} \tag{32}$$

$$8\pi\rho = \frac{2m}{r_3}\left(\frac{1}{r^2}\right) = \frac{(1-n^2)}{(2-n^2)}\left(\frac{1}{r^2}\right) \tag{33}$$

where m = total mass of the sphere

$$e^{\nu} = (D_1 r^{1+n} + D_2 r^{1-n})^2 \tag{34}$$

$$= \frac{1}{16n^2(2-n^2)}\left((1+n)^2\left(\frac{r}{r_3}\right)^{1-n} - (1-n)^2\left(\frac{r}{r_3}\right)^{1+n}\right)^2 \tag{35}$$

$$8\pi P = \frac{(1-n^2)^2}{(2-n^2)r^2}\left(\frac{(r/r_3)^{-n} - (r/r_3)^{n}}{(1+n)^2(r/r_3)^{-n} - (1-n)^2(r/r_3)^{n}}\right). \tag{36}$$

Middle region:

$$e^{-\lambda} = 1 - \frac{8\pi\rho' K^2}{15}(5x-3x^2) \qquad K \geqslant r_2 \tag{37}$$

$$\frac{8\pi\rho' r_2^2}{3}\left(1-\frac{3r_2^2}{5K^2}\right) = \frac{2m}{r_3} = \left(\frac{1-n^2}{2-n^2}\right) \tag{38}$$

$$e^{\nu} = (C_1\cos\tfrac{1}{2}w + C_2\sin\tfrac{1}{2}w)^2 \tag{39}$$

$$8\pi P = M\left(\frac{C_2 - C_1\tan\frac{1}{2}w}{C_1 + C_2\tan\frac{1}{2}w}\right) - N \tag{40}$$

where M, N and w are similar to those appearing in equation (13). The constants C_1, and C_2 can be evaluated from equations

$$M_2\left(\frac{C_2-C_1\tan\frac{1}{2}w_2}{C_1+C_2\tan\frac{1}{2}w_2}\right)-N_2=\frac{(1-n^2)^2}{(2-n^2)r_2{}^2}\left(\frac{(r_2/r_3)^{-n}-(r_2/r_3)^n}{(1+n)^2(r_2/r_3)^{-n}-(1-n)^2(r_2/r_3)^n}\right)$$

$$C_1\cos\tfrac{1}{2}w_2+C_2\sin\tfrac{1}{2}w_2=\frac{1}{4n(2-n^2)^{1/2}}\left((1+n)^2\left(\frac{r_2}{r_3}\right)^{1-n}-(1-n)^2\left(\frac{r_2}{r_3}\right)^{1+n}\right).$$

M_2, N_2 and w_2 are the respective values of M, N and w when $x=r_2{}^2/K^2$

The central core $0\leqslant r\leqslant r_1$: The density is constant

$$e^{-\lambda}=1-r^2/R^2\qquad R^2=3/8\pi\rho_c$$

$$\frac{8\pi\rho_c r_1{}^2}{3}=\frac{8\pi\rho' r_2{}^2}{15}(5-3r_1{}^2/K^2)$$

$$\rho_c=\rho'(1-3r_1{}^2/5K^2)$$

$$e^{\nu}=\{A-B(1-r^2/R^2)^{1/2}\}^2$$

$$8\pi P=\frac{1}{R^2}\left(\frac{3B(1-r^2/R^2)^{1/2}-A}{A-B(1-r^2/R^2)^{1/2}}\right)$$

where

$$A=\tfrac{1}{2}e^{\nu_1/2}(3+8\pi P_1R^2)$$

$$B=\tfrac{1}{2}e^{\nu_1/2}(1+8\pi P_1R^2)(1-r_1{}^2/R^2)^{-1/2}$$

$$e^{\nu_1/2}=C_1\cos\tfrac{1}{2}w_1+C_2\sin\tfrac{1}{2}w_1$$

$$8\pi P_1=M_1\left(\frac{C_2-C_1\tan\frac{1}{2}w_1}{C_1+C_2\tan\frac{1}{2}w_1}\right)-N_1.$$

3. Discussions

The pressure density restrictions in (vi) limit the values of parameters further. If we have at $r=0$

$$P\leqslant\rho_c/3\qquad\text{we get}\qquad A\geqslant 2B$$

$$P\leqslant\rho_c\qquad\text{we get}\qquad A\geqslant 3B/2$$

in order that the pressure should remain non-negative we must further restrict $A<3B$.

The values of e^{λ}, e^{ν} and P have been shown to be continuous at all the boundaries, but the densities at the boundaries change from one region to another.

In Part (1) of the solutions we have

at $r=r_1$: $\quad\rho\,(\text{middle})_{r=r_1}=\tfrac{1}{3}\rho_c$

at $r=r_2$: $\quad\rho\,(\text{middle})_{r=r_2}=\dfrac{1}{3}\dfrac{r_1{}^2}{r_2{}^2}\rho_c=\tfrac{1}{3}(1-3r_2{}^2/5K^2)\rho'$

$$\rho\,(\text{outer})_{r=r_2}=\rho'\left(1-\frac{r_2{}^2}{K^2}\right)=\rho_c\frac{r_1{}^2}{r_2{}^2}\left(\frac{1-r_2{}^2/K^2}{1-3r_2{}^2/5K^2}\right)$$

at $r = r_3$: $\rho_{r=r_3}$ (surface) $= \rho_c r_1^2(1-r_3^2/K^2)/r_2^2(1-3r_2^2/5K^2)$.

In Part (2) of the solutions we have

at $r = r_1$: $\rho_{r=r_1}$ (middle) $= \rho_c(1-r_1^2/K^2)/(1-3r_1^2/5K^2)$

at $r = r_2$: $\rho_{r=r_2}$ (middle) $= \rho'(1-r_2^2/K^2) = \rho_c \dfrac{(1-r_2^2/K^2)}{(1-3r_1^2/5K^2)}$

$\rho_{r=r_2}$ (outer) $= \frac{1}{3}\rho_c(1-3r_2^2/5K^2)/(1-3r_1^2/5K^2)$

at $r = r_3$: $\rho_{r=r_3}$ (surface) $= \frac{1}{3}\rho_c r_2^2(1-3r_2^2/5K^2)/r_3^2(1-3r_1^2/5K^2)$.

We see that the densities at the boundaries in different regions depend upon the choice of various radii.

For the values of the parameter n in the region of inverse square variation of density we have

$$C = (1-n^2)/(2-n^2) \qquad \text{or} \qquad n^2 = (1-2C)/(1-C)$$

n is always less than 1 and positive. From the expression for e^ν we see that physically possible solutions exist when n is real and non-negative. Hence

$$C \leqslant \tfrac{1}{2} \qquad \text{or} \qquad 2m'/r_2 \leqslant \tfrac{1}{2}$$

in part (1) where m' is the mass contained within the radius r_2, and

$$2m/r_3 \leqslant \tfrac{1}{2}$$

in part (2) where m is the total mass of the sphere.

The results obtained in this paper can be applied to various problems by proper choice of the parameters involved.

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